EXPONENTIAL ERGODICITY OF NON-LIPSCHITZ MULTIVALUED STOCHASTIC DIFFERENTIAL EQUATIONS*

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Abstract

We prove the exponential ergodicity of the transition probabilities of solutions to elliptic multivalued stochastic differential equations.

Résumé

On prouve l'ergodicité exponentielle des probabilités de transition des equations différentielles stochastiques elliptiques.

1. Introduction and Preliminaries

Consider the following stochastic differential equation:

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad X_0 = x, \tag{1}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^n$ are continuous functions, $(W_t)_{t\geqslant 0}$ is an n-dimensional standard Brownian motion defined on some complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. When σ is a uniformly elliptic square matrix and σ and b satisfy some regular conditions (more precisely, (H1), (H2) and (H4) below), it is recently proved in [6] that the solution is exponentially ergodic.

On the other hand, under the same uniform elliptic assumption and an additional one that σ and b are C_b^2, Cépa and Jacquot proved in [2] the ergodicity for the solution of the following stochastic variational inequality (SVI in short):

$$\mathrm{d}X_t + \partial\varphi(X_t) \ni b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad X_0 = x \in \overline{\mathrm{Dom}(\varphi)}, \tag{2}$$

where $\partial\varphi$ is the sub-differential of some convex function φ with a compact domain $\mathrm{Dom}(\varphi) = \{x : \varphi(x) < \infty\}$.

A common drawback of the above two papers is the *uniform elliptic assumption* of the diffusion coefficients. The purpose of the present paper is to remove this assumption and instead assume only the *ellipticity*. Our main result as stated in Theorem 2.1 below unifies and improves the main results of both of [6] and [2]. In particular, our result applies to stochastic variational inequalities defined on non-compact domains. Furthermore, we do not need to assume that the diffusion matrix is square and our method even works for general multivalued stochastic differential equations (MSDEs in abbreviation):

$$\mathrm{d}X_t + A(X_t) \ni b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad X_0 = x \in \overline{D(A)}, \tag{3}$$

where A is a multivalued maximal monotone operator on $\mathbb{R}^d$ with $\mathrm{Int}(D(A)) \neq \emptyset$.

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Now we introduce notions and notations. Given an operator A from $\mathbb{R}^d$ to $2^{\mathbb{R}^d}$, define:

$$\begin{aligned} D(A) &:= \{x \in \mathbb{R}^d : A(x) \neq \emptyset\}, \\ \mathrm{Gr}(A) &:= \{(x,y) \in \mathbb{R}^{2d} : x \in \mathbb{R}^d, y \in A(x)\}. \end{aligned}$$

Then A is called monotone if $\langle y_1 - y_2, x_1 - x_2 \rangle \geqslant 0$ for any $(x_1,y_1),(x_2,y_2) \in \mathrm{Gr}(A)$, and A is called maximal monotone if

$$(x_1,y_1) \in \mathrm{Gr}(A) \Leftrightarrow \langle y_1 - y_2, x_1 - x_2 \rangle \geqslant 0, \quad \forall (x_2,y_2) \in \mathrm{Gr}(A).$$

Definition 1.1. *A pair of continuous and $(\mathcal{F}_t)$-adapted processes (X,K) is called a solution of (3) if*

(i) $X_0 = x_0$, $X_t \in \overline{D(A)}$ a.s.;

(ii) K is of locally finite variation and $K_0 = 0$ a.s.;

(iii) $\mathrm{d}X_t = b(t,X_t)\mathrm{d}t + \sigma(t,X_t)\mathrm{d}W_t - \mathrm{d}K_t$, $0 \leqslant t < \infty$, a.s.;

(iv) For any continuous and $(\mathcal{F}_t)-$adapted functions (α,β) with $(\alpha_t,\beta_t) \in \mathrm{Gr}(A)$, $\forall t \in [0,+\infty)$, the measure $\langle X_t - \alpha_t, \mathrm{d}K_t - \beta_t \mathrm{d}t \rangle$ is positive.

We make the following assumptions:

(H1) (Monotonicity) There exists $\lambda_0 \in \mathbb{R}$ such that for all $x,y \in \mathbb{R}^d$

$$2\langle x-y, b(x)-b(y)\rangle + \|\sigma(x)-\sigma(y)\|_{\mathrm{HS}}^2 \leqslant \lambda_0 |x-y|^2 (1 \vee \log|x-y|^{-1}).$$

(H2) (Growth of σ) There exists $\lambda_1 > 0$ such that for all $x \in \mathbb{R}^d$

$$\|\sigma(x)\|_{\mathrm{HS}} \leqslant \lambda_1 (1+|x|).$$

(H3) (Ellipticity of σ)

$$\sigma\sigma^*(x) > 0, \ \forall x \in \mathbb{R}^d.$$

(H4) (One side growth of b) There exist a $p \geqslant 2$ and constants $\lambda_3 > 0, \lambda_4 \geqslant 0$ such that for all $x \in \mathbb{R}^d$

$$2\langle x, b(x)\rangle + \|\sigma(x)\|_{\mathrm{HS}}^2 \leqslant -\lambda_3 |x|^p + \lambda_4.$$

Theorem 1.2. *Assume* **(H1)** *and* **(H2)** *hold. Then (3) has a unique strong solution.*

Proof. The existence of a weak solution is proved in [2] and the pathwise uniqueness can be proved in a more or less standard way using a version of Bihari inequality (see [5]). Finally by Yamada-Watanabe's theorem the existence of a unique strong solution follows. □

Let $\{X_t(x), t \geqslant 0, x \in \mathbb{E}\}$ denote the unique solution to (3). It is obviously a Markow family and its transition semigroup and transition probability are defined respectively as:

$$P_t f(x_0) := \mathbf{E} f(X_t(x_0)), \ t > 0, \ f \in B_b(\mathbb{R}^d)$$

and

$$P_t(x_0, E) := \mathbf{P}(X_t(x_0) \in E),$$

where $x_0 \in \mathbb{E}$ and $B_b(\mathbb{R}^d)$ denotes the set of all bounded measurable functions on $\mathbb{R}^d$. For general notions (e.g., strong Feller property, irreducibility, ergodicity, etc) concerning Markov semigroups, we refer to [2, 6].

2. Main Result

Now we state the main result of the paper.

Theorem 2.1. *Assume* **(H1)-(H3)**. *Then the transition probability P_t of the solution to (3) is irreducible and strong Feller. If in addition,* **(H4)** *holds, then there exists a unique invariant probability measure μ of P_t having full support in $\overline{D(A)}$ such that*

(i) If $p \geqslant 2$ in **(H4)**, *then for all $t > 0$ and $x_0 \in \overline{D(A)}$, μ is equivalent to $P_t(x_0, \cdot)$, and*
$$\lim_{t\to\infty} \|P_t(x_0, \cdot) - \mu\|_{\mathrm{Var}} = 0,$$
where $\|\cdot\|_{\mathrm{Var}}$ denotes the total variation of a signed measure.

(ii) If $p > 2$ in **(H4)**, *then for some $\alpha, C > 0$ independent of x_0 and t,*
$$\|P_t(x_0, \cdot) - \mu\|_{\mathrm{Var}} \leqslant C \cdot e^{-\alpha t}.$$
Moreover, for any $q > 1$ and each $\varphi \in L^q(\overline{D(A)}, \mu)$
$$\|P_t\varphi - \mu(\varphi)\|_q \leqslant C_q \cdot e^{-\alpha t/q}\|\varphi\|_q, \quad \forall t > 0,$$
where α is the same as above and $\mu(\varphi) := \int_{\overline{D(A)}} \varphi(x)\mu(\mathrm{d}x)$. In particular, let L_q be the generator of P_t in $L^q(\overline{D(A)}, \mu)$. Then L_q has a spectral gap (greater than α/q) in $L^q(\overline{D(A)}, \mu)$.

The proof consists in proving the irreducibility and strong Feller property.

2.1. Irreducibility.

Lemma 2.2. *Suppose $y_0 \in Int(D(A))$, $m > 0$, and Y_t is the solution to the following MSDE:*
$$\mathrm{d}Y_t + A(Y_t)\mathrm{d}t \ni -m(Y_t - y_0)\mathrm{d}t + \sigma(Y_t)\mathrm{d}W_t,\ Y_0 = x_0,$$
where σ is the diffusion coefficient of (3). Then under **(H1)** *and* **(H2)** *we have*
$$\mathbf{E}|Y_t - y_0|^2 \leqslant e^{-C(m)t}|x_0 - y_0|^2 + \frac{C_0}{C(m)},$$
where $C(m) = 2(m - 2\lambda_1^2 - 1/2)$ and $C_0 = 2\lambda_1^2(1 + 2|y_0|^2) + |A^\circ(y_0)|^2$. Here A° is the minimal section of A and $|A^\circ(y_0)| < +\infty$ because $y_0 \in Int(D(A))$ (see [1]).

Proof. The proof is adapted from [2]. Consider the solution Y_t^n to the following equation:
$$\mathrm{d}Y_t^n + A_n(Y_t^n)\mathrm{d}t = -m(Y_t^n - y_0)\mathrm{d}t + \sigma(Y_t^n)\mathrm{d}W_t,\ Y_0^n = x_0$$
where A_n is the Yosida approximation of A. From [1] we know that A_n is monotone, single-valued and $|A_n(x)| \nearrow |A^\circ(x)|$ if $x \in D(A)$, where A° is the minimal section of A. Moreover, since the law of Y_t^n converges to that of Y_t, it is enough to prove the inequality for Y_t^n. Hence by **(H2)**
$$\begin{aligned}
& -2m|x - y_0|^2 + \|\sigma(x)\|_{\mathrm{HS}}^2 - 2\langle A_n(x), x - y_0\rangle \\
\leqslant\ & -2m|x - y_0|^2 + \lambda_1^2(1 + |x|)^2 - 2\langle A_n(x) - A_n(y_0), x - y_0\rangle - 2\langle A_n(y_0), x - y_0\rangle \\
\leqslant\ & -2m|x - y_0|^2 + \lambda_1^2(1 + |x|)^2 + |x - y_0|^2 + |A^\circ(y_0)|^2 \\
\leqslant\ & -2m|x - y_0|^2 + 2\lambda_1^2(1 + 2|x - y_0|^2 + 2|y_0|^2) + |x - y_0|^2 + |A^\circ(y_0)|^2 \\
=\ & -C(m)|x - y_0|^2 + C_0,
\end{aligned}$$
Thus, by Itô's formula we have
$$\frac{d}{\mathrm{d}t}\mathbf{E}|Y_t^n - y_0|^2 = -2\mathbf{E}(\langle Y_t^n - y_0, A_n(Y_t^n)\rangle) + \mathbf{E}[\mathrm{Tr}(\sigma\sigma^*(Y_t^n))] - 2m\mathbf{E}|Y_t^n - y_0|^2$$

$$\leqslant \quad -C(m)\mathbf{E}|Y_t^n - y_0|^2 + C_0.$$

Therefore

$$\mathbf{E}|Y_t^n - y_0|^2 \leqslant e^{-C(m)t}|x_0 - y_0|^2 + \frac{C_0}{C(m)}.$$

$\square$

Proposition 2.3. *Under* **(H1)**-**(H3)***, the transition probability* P_t *is irreducible.*

Proof. It suffices to prove that for any $x_0 \in \overline{D(A)}$, $T > 0$, $y_0 \in \mathrm{Int}(D(A))$ and $a > 0$,

$$P_T(x_0, B(y_0, a)) = \mathbf{P}(X_T(x_0) \in B(y_0, a)) = \mathbf{P}(|X_T(x_0) - y_0| \leqslant a) > 0,$$

or equivalently:

$$\mathbf{P}(|X_T(x_0) - y_0| > a) < 1.$$

Fix a, T and y_0. By Lemma 2.2 and Chebyshev's inequality, we can choose an m large enough such that, denoting by $(Y_t, \tilde{K}_t)$ the unique solution to

$$\mathrm{d}Y_t + A(Y_t)\mathrm{d}t \ni -m(Y_t - y_0)\mathrm{d}t + \sigma(Y_t)\mathrm{d}W_t, \ Y_0 = x_0 \in \overline{D(A)}, \tag{4}$$

we have

$$\mathbf{P}(|Y_T(x_0) - y_0| > a) \leqslant \left(e^{-C(m)T}|x_0 - y_0|^2 + \frac{C_0}{C(m)}\right)/a^2 < 1. \tag{5}$$

Set

$$\tau_N := \inf\{t : |Y_t| \geqslant N\}.$$

Note that by [2]

$$\mathbf{E}\left[\sup_{t\in[0,T]} |Y_t(x_0)|\right] \leqslant C$$

for some constant C depending on x_0, y_0, λ_1, m and T. Thus we may fix an N so that

$$\mathbf{P}(\tau_N \leqslant T) + \mathbf{P}(|Y_T(x_0) - y_0| > a) < 1. \tag{6}$$

Define

$$U_t := \sigma(Y_t)^*[\sigma(Y_t)\sigma(Y_t)^*]^{-1}(-m(Y_t - y_0) - b(Y_t))$$

and

$$Z_T = \exp\left(\int_0^{T\wedge\tau_N} U_s \mathrm{d}W_s - \frac{1}{2}\int_0^{T\wedge\tau_N} |U_s|^2 \mathrm{d}s\right).$$

Since $|U_{t\wedge\tau_N}|^2$ is bounded, $\mathbf{E}[Z_T] = 1$ by Novikov's criteria.

By Girsanov's theorem, $W_t^* := W_t + V_t$ is a Q-Brownian motion, where

$$V_t := \int_0^{t\wedge\tau_N} U_s \mathrm{d}s, \quad Q := Z_T\mathbf{P}.$$

By (6) we have

$$Q(\{\tau_N \leqslant T\} \cup \{|Y_T(x_0) - y_0| > a\}) < 1. \tag{7}$$

Note that the solution $(Y_t, \tilde{K}_t)$ of (4) also solves the MSDE below

$$Y_t + \int_0^t A(Y_s)\mathrm{d}s \ni \int_0^t \sigma(Y_s)\mathrm{d}W_s^* + \int_0^{t\wedge\tau_N} b(Y_s)\mathrm{d}s - \int_{t\wedge\tau_N}^t m(Y_s - y_0)\mathrm{d}s.$$

Set

$$\theta_N := \inf\{t : |X_t| \geqslant N\}.$$

Then the uniqueness in distribution for (4) yields that the law of $\{(X_t\mathbf{1}_{\{\theta_N\geqslant T\}})_{t\in[0,T]},\theta_N\}$ under $\mathbf{P}$ is the same as that of $\{(Y_t\mathbf{1}_{\{\tau_N\geqslant T\}})_{t\in[0,T]},\tau_N\}$ under Q. Hence

$$\begin{aligned}\mathbf{P}(|X_T(x_0)-y_0|>a) &\leqslant \mathbf{P}(\{\theta_N\leqslant T\}\cup\{\theta_N\geqslant T,|X_T(x_0)-y_0|>a\})\\ &= Q(\{\tau_N\leqslant T\}\cup\{\tau_N\geqslant T,|Y_T(x_0)-y_0|>a\})\\ &\leqslant Q(\{\tau_N\leqslant T\}\cup\{|Y_T(x_0)-y_0|>a\})<1.\end{aligned}$$

□

2.2. **Strong Feller Property.** The proof of the following lemma is plain by using Kolmogorov's lemma on path regularity of stochastic processes.

Lemma 2.4. *Denote by $(X_t(x),K_t(x))$ the solution of (3) with inital value x. Then for any $p>d$, there exists $t_p>0$ such that for all $r>0$*

$$\mathbf{E}\left[\sup_{x\in D_r,s\leqslant t_p}|X_s(x)|^p\right]<\infty,$$

where $D_r:=\overline{D(A)}\cap\{|x|\leqslant r\}$.

Proposition 2.5. *Under* **(H**1**)**-**(H**3**)**, *the semigroup P_t is strong Feller.*

Proof. We divide the proof into two steps.

Step 1: Assume that there exists a $\lambda_2>0$ such that $\|[\sigma^*\sigma]^{-1}\|_{\mathrm{HS}}\leqslant\lambda_2$. Consider the following drift transformed MSDE:

$$\begin{cases}\mathrm{d}Y_t+A(Y_t)\mathrm{d}t\ni b(Y_t)\mathrm{d}t+\sigma(Y_t)\mathrm{d}W_t+|x_0-y_0|^\alpha\frac{X_t-Y_t}{|X_t-Y_t|}\cdot 1_{\{X_t\neq Y_t\}}\cdot 1_{\{t<\tau\}}\mathrm{d}t,\\ Y_0\ =y_0\in\overline{D(A)},\end{cases}\tag{8}$$

where $\alpha\in(0,1)$, X_t is the solution to (3) and τ is the coupling time given by

$$\tau:=\inf\{t>0:|X_t-Y_t|=0\}.$$

An argument similar to [6] allows to prove it admits a unique solution.

For $T>0$ define

$$U_T:=\exp\left[\int_0^{T\wedge\tau}\langle\mathrm{d}W_s,H(X_s,Y_s)\rangle-\frac{1}{2}\int_0^{T\wedge\tau}|H(X_s,Y_s)|^2\mathrm{d}s\right]$$

and

$$\tilde{W}_t:=W_t+\int_0^{t\wedge\tau}H(X_s,Y_s)\mathrm{d}s,$$

where

$$H(x,y):=|x_0-y_0|^\alpha\cdot\sigma^*(y)[\sigma\sigma^*(y)]^{-1}\frac{x-y}{|x-y|}.$$

Since $\|[\sigma\sigma^*(y)]^{-1}\|_{\mathrm{HS}}\leqslant\lambda_2$, we have

$$|H(x,y)|^2\leqslant\lambda_2\cdot|x_0-y_0|^{2\alpha}.$$

Thus,

$$\mathbf{E}U_T=1\quad\text{and}\quad\mathbf{E}U_T^2\leqslant\exp\left[\lambda_2T\cdot|x_0-y_0|^{2\alpha}\right].$$

By the elementary inequality $e^r-1\leqslant re^r$ for $r\geqslant0$, we have for any $|x_0-y_0|\leqslant\eta$,

$$\begin{aligned}(\mathbf{E}|1-U_T|)^2 &\leqslant \mathbf{E}|1-U_T|^2=\mathbf{E}U_T^2-1\\ &\leqslant \exp\left[\lambda_2T\cdot|x_0-y_0|^{2\alpha}\right]-1\\ &\leqslant C_{T,\lambda_2,\eta}\cdot|x_0-y_0|^{2\alpha}\end{aligned}\tag{9}$$

and

$$\begin{aligned}\left(\mathbf{E}\left[(1+U_T)1_{\{\tau\geqslant T\}}\right]\right)^2 &\leqslant (3+\mathbf{E}U_T^2)\cdot\mathbf{P}(\tau\geqslant T)\\ &\leqslant C_{T,\lambda_2,\eta}\cdot\mathbf{P}((2T)\wedge\tau\geqslant T)\\ &\leqslant C_{T,\lambda_2,\eta}\cdot\mathbf{E}((2T)\wedge\tau)/T. \end{aligned}\tag{10}$$

First applying Itô's formula to $\sqrt{|Z_{t\wedge\tau}|^2+\varepsilon}$ where $Z_s := X_s - Y_s$, then letting $\varepsilon\downarrow 0$, and finally taking expectation, we have by **(H1)**,

$$\mathbf{E}|X_{t\wedge\tau}-Y_{t\wedge\tau}|\leqslant |x_0-y_0|-|x_0-y_0|^{\alpha}\cdot\mathbf{E}(t\wedge\tau)+\frac{\lambda_0}{2}\int_0^t\rho_\eta(\mathbf{E}|X_{s\wedge\tau}-Y_{s\wedge\tau}|)\mathrm{d}s,$$

which implies by Bihari inequality that for any $t>0$ and $|x_0-y_0|<\eta$

$$\mathbf{E}|X_{t\wedge\tau}-Y_{t\wedge\tau}|\leqslant |x_0-y_0|^{\exp\{-\lambda_0 t/2\}}$$

and thus

$$\mathbf{E}(t\wedge\tau)\leqslant |x_0-y_0|^{1-\alpha}+\frac{\lambda_0 t}{2}\rho_\eta(|x_0-y_0|^{\exp\{-\lambda_0 t/2\}})\cdot|x_0-y_0|^{-\alpha}.\tag{11}$$

Taking $\alpha=\exp\{-\lambda_0 T\}/2$, there exists an $0<\eta'<\eta$ such that for any $|x_0-y_0|<\eta'$

$$\mathbf{E}((2T)\wedge\tau)\leqslant C_{T,\lambda_0,\eta'}\cdot|x_0-y_0|^{\exp\{-\lambda_0 T\}/2}.\tag{12}$$

But by Girsanov's theorem, $(\tilde W_t)_{t\in[0,T]}$ is still a n-dimensional Brownian motion under the new probability measure $U_T\cdot\mathbf{P}$. Note that $(Y_t,\tilde K_t)$ also solves

$$\mathrm{d}Y_t+A(Y_t)\mathrm{d}t\ni b(Y_t)\mathrm{d}t+\sigma(Y_t)\mathrm{d}\tilde W_t,\quad Y_0=y_0.$$

So, the law of $X_T(y_0)$ under $\mathbf{P}$ is the same as that of $Y_T(y_0)$ under $U_T\cdot\mathbf{P}$. Thus by (9), (10) and (12), for any $f\in B_b(\mathbb{R}^d)$,

$$\begin{aligned}&|P_Tf(x_0)-P_Tf(y_0)|=|\mathbf{E}(f(X_T(x_0))-U_T\cdot f(Y_T(y_0)))|\\ \leqslant\ &\mathbf{E}\left|(1-U_T)\cdot f(X_T(x_0))\cdot 1_{\{\tau\leqslant T\}}\right|+\mathbf{E}\left|(f(X_T(x_0))-U_T\cdot f(Y_T(y_0)))\cdot 1_{\{\tau>T\}}\right|\\ \leqslant\ &\|f\|_0\cdot\mathbf{E}|1-U_T|+\|f\|_0\cdot\mathbf{E}\left[(1+U_T)1_{\{\tau>T\}}\right]\\ \leqslant\ &C_{T,\lambda_0,\lambda_2,\eta}\cdot\|f\|_0\cdot|x_0-y_0|^{\exp\{-\lambda_0 T\}/4}.\end{aligned}$$

Step 2: Now we prove the proposition under **(H3)**. By the Markov property of the solution, we only need to prove that for every $f\in B_b(\mathbb{R}^d)$, $x\mapsto P_tf(x)$ is continuous on D_r for all $t\leqslant t_p$, $p>d$ where p and t_p are specified in Lemma 2.4. Set

$$c_0:=\|f\|_\infty$$

and

$$\tau:=\inf\left\{t>0:\sup_{x\in D_r}|X_t(x)|>N\right\}.$$

Let $\varepsilon>0$ be given. For $t\leqslant t_p$, by Lemma 2.4 and Chebyshev inequality, there exists $N>r$ such that

$$\mathbf{P}(\tau\leqslant T)=\mathbf{P}\left(\sup_{x\in D_r,t\leqslant t_p}|X_t(x)|>N\right)\leqslant\mathbf{E}\left[\sup_{x\in D_r,t\leqslant t_p}|X_t(x)|^p\right]/N^p<\varepsilon.\tag{13}$$

Define

$$\tilde\sigma(x):=\sigma(x),\ \forall|x|\leqslant N.$$

Extend $\tilde{\sigma}$ to the whole $\mathbb{R}^d$ such that it satisfies the condition **(H1)** to **(H3)**. Denote by $\tilde{X}_t(x)$ the solution to (3) with σ replaced by $\tilde{\sigma}$. By Step 1, there exists a $\delta > 0$ such that if $|x-y| < \delta$ and $x, y \in D_r$,

$$|\mathbf{E}[f(\tilde{X}_t(x))] - \mathbf{E}[f(\tilde{X}_t(y))]| < \varepsilon. \tag{14}$$

Hence

$$\begin{aligned} & |\mathbf{E}[f(X_t(x))] - \mathbf{E}[f(X_t(y))]| \\ \leqslant\ & |\mathbf{E}[(f(X_t(x)) - f(X_t(y)))1_{(\tau > T)}]| + |\mathbf{E}[(f(X_t(x)) - f(X_t(y)))1_{(\tau \leqslant T)}]| \\ \leqslant\ & |\mathbf{E}[(f(\tilde{X}_t(x)) - f(\tilde{X}_t(y)))1_{(\tau > T)}]| + 2c_0\varepsilon \\ \leqslant\ & |\mathbf{E}[f(\tilde{X}_t(x)) - f(\tilde{X}_t(y))]| + |\mathbf{E}[(f(\tilde{X}_t(x)) - f(\tilde{X}_t(y)))1_{(\tau \leqslant T)}]| + 2c_0\varepsilon \\ \leqslant\ & (1 + 4c_0)\varepsilon. \end{aligned}$$

□

Now we are in a position to complete the proof of Theorem 2.1.

Proof. (i) By Itô's formula and **(H**4**)**, we get

$$\begin{aligned} \mathbf{E}|X_t|^2 &= |x_0|^2 + 2\int_0^t \mathbf{E}\langle X_s, b(X_s)\rangle \mathrm{d}s - 2\int_0^t \mathbf{E}\langle X_s, \mathrm{d}K_s\rangle + \int_0^t \mathbf{E}\|\sigma(X_s)\|_{\mathrm{HS}}^2 \mathrm{d}s \\ &\leqslant |x_0|^2 + \int_0^t \mathbf{E}(-\lambda_3|X_s|^p + \lambda_4)\mathrm{d}s. \end{aligned}$$

Taking derivatives with respect to t and using Hölder's inequality give

$$\frac{\mathrm{d}\mathbf{E}|X_t|^2}{\mathrm{d}t} \leqslant -\lambda_3\mathbf{E}|X_t|^p + \lambda_4 \leqslant -\lambda_3(\mathbf{E}|X_t|^2)^{p/2} + \lambda_4.$$

Since $\lambda_3 > 0$ we have for all $t > 0$,

$$\frac{1}{t}\int_0^t \mathbf{E}|X_s|^2 \mathrm{d}s \leqslant \lambda_4/\lambda_3.$$

Therefore by Krylov-Bogoliubov's method (see [3]), there exists an invariant probability measure μ. As we have just proved, P_t is strong Feller and irreducible. Then, again by [3], μ is equivalent to each $P_t(x, \cdot)$ with $x \in \overline{D(A)}, t > 0$ and consequently (i) holds.

(ii) If $p > 2$, consider the following ODE:

$$f'(x) = -\lambda_3 f(x)^{p/2} + \lambda_4,\ f(0) = |x_0|^2.$$

By the comparison theorem (cf. [3]), there exists some $C > 0$ such that

$$\mathbf{E}|X_t|^2 \leqslant f(t) \leqslant C(1 + t^{2/(2-p)}).$$

We also have

$$\inf_{x_0 \in B(0,r)} P_t(x_0, B(0,a)) > 0,\ \forall r, a > 0,\ t > 0$$

because of the strong Feller property and irreducibility. Therefore (ii) holds due to Theorem 2.5 (b) and Theorem 2.7 in [4]. □

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